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TECHNICAL NOTE

Closed-form expressions to Gompertz-Makeham life expectancies: a historical note

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Results well known in the actuarial community about closed-form expressions to Gompertz and Gompertz-Makeham life expectancies for a person aged x are still being independently rediscovered to this day. This note seeks to acknowledge previous results about closed-form expressions to Gompertz-Makeham life expectancies, especially in the actuarial science field, hoping to stimulate interdisciplinarity and provide the background for further developments, especially since the derivation of closed-form expressions for life expectancy (and annuities) based on particular mortality laws are matters of interest for multiple fields such as actuarial science, biology, demography, statistics among others.

Keywords: Gompertz-Makeham mortality law. Life expectancy. Actuarial science. Annuities. Frailty. Gamma-Gompertz model.

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Introduction

Actuarial science started to become a formal discipline in the 17th century, based on three pillars: compound interest, probability theory and empirical life tables. With those tools, British mathematician Edmund Halley published in 1693 a paper describing the construction of an empirical life table for the city of Breslau, and also provided a method for pricing life annuities (HABERMAN, 1996). Halley's contribution is a landmark in the history of actuarial science, and it indicates that the valuation of life annuities was one of the first issues to be addressed by the field.

In 1725, seven years after the publication of his Doctrine of Chances, de Moivre published the book Annuities on Lives where, from age 12 to 86, he fitted a straight line through Halley's life table (TEUGELS, 2006; FORFAR, 2006), which allowed him to compute the value of a life annuity for a person aged x as a linear function of an annuity-certain (HABERMAN, 1996). This attempt to summarize mortality as a mathematical expression is now known as the de Moivre mortality law.

Ever since de Moivre, attempts to summarize empirical observations on mortality as a mathematical formula have drawn the attention of actuaries and other researchers for theoretical and practical reasons. For the actuarial community, as stated by Rietz (1921), calculations of annuities and life insurance were very laborious in the early days, which stimulated the development of mortality laws from which the expected present value of annuities and life insurance could be more easily computed. In this context, the seminal contribution of British actuary Benjamin Gompertz (1825) deserves a special place in actuarial history.

As a practicing actuary, Gompertz was interested in the problem of computing premiums for insurance products, which made life tables fundamental tools. However, Gompertz's interests were not limited to practical actuarial problems. He also intended to understand the patterns of the dying-out process of human populations during a significant portion of their lives (OLSHANSKY; CARNES, 1997). To that end, he analyzed death records from England, France and Sweden in the 19th century, for people between 20 and 60 years old (OLSHANSKY, 2010), and concluded that death rates increased exponentially with age. According to Olshansky (2010), those observations led Gompertz to believe he had discovered a general law of human mortality comparable in importance to Newton's law of gravity.

William Makeham (1860, 1867), another British actuary, provided his first and most famous revision of the Gompertz law adding to Gompertz's formula a constant independent of age, which partitioned mortality into biological and non-biological elements (OLSHANSKY; CARNES, 1997; FORFAR, 2006). This formula is now known as the Gompertz-Makeham (or simply Makeham) mortality law. Throughout the 20th century and to this day, Gompertz and Makeham's contributions to actuarial science have inspired scientists in a variety of fields, such as Demography, Biodemography, Evolutionary Biology (OLSHANSKY; CARNES,

1997; OLSHANSKY, 2010; MISSOV; FINKELSTEIN, 2011; VAUPEL; MISSOV, 2014; MISSOV; NÉMETH, 2016; MISSOV; NÉMETH; DAŃKO, 2016; NÉMETH; MISSOV, 2018), Statistics and Applied Mathematics (JODRÁ, 2009, 2013; DEY; MOALA; KUMAR, 2018; BÖHNSTEDT; GAMPE, 2019; SALINARI; DE SANTIS, 2020), Business (BEMMAOR; GLADY, 2012; MILEVSKY; SALISBURY; ALEXANDER, 2016) and, evidently, Actuarial Science (PITACCO, 2004; BUTT; HABERMAN, 2004; LI *et al.*, 2021), among others.

A particular topic that has been drawing attention of researchers from different fields to this day is the desire to provide analytical expression (using special mathematical functions) to Gompertz and Gompertz-Makeham life expectancies for the case of homogeneous or gamma-heterogeneous population for a person aged x, as can be seen, for example, in the works of Scarpello, Ritelli and Spelta (2006), Dey, Moala and Kumar (2018), Castellares, Patrício and Lemonte (2020) and Patrício (2020) in the statistical field, Missov and Lenart (2013) and Castellares *et al.* (2020), in the theoretical biology field, Bemmaor and Glady (2012) and Adler (2022) in the management field, and Missov (2013, 2021), Missov and Lenart (2011), Missov, Lenart and Vaupel (2012) in the demographic field.

We recognize the difficulty to define the "maternity or paternity" of an academic achievement clearly, especially in areas with long academic tradition and vast body of knowledge production, with results published in different languages and/or with very restricted circulation, sometimes only available (or known) from secondary sources or brief references. Additionally, it is not unusual to find researchers from different fields working and providing independent contributions to the same problemin a parallel manner. Nevertheless, in this note, we intend to provide some recognition to previous results about closed-form expressions to Gompertz-Makeham life expectancies, especially, as stated, in the actuarial science field, also pointing new contributions to discussions initiated by the actuarial community.

At this point, it is worth mentioning that, throughout the text, original notations of some expressions were adapted to avoid confusion with symbols reserved to other formulas. Moreover, since among the recent rediscoveries of closed form expression to Gompertz-Makeham life expectancies for a person aged x the interesting work of Castellares *et al.* (2020) is the most complete one and also provides new contributions to the subject, we compare previous results in the literature with those of Castellares *et al.* (2020), which allows for a clearer view of previous contributions made in actuarial science and the new advances being developed.

Analytic expressions to life annuities based on the Gompertz-Makeham mortality law

Classical actuarial mathematics textbooks written in English, such as Jordan (1967), Bowers *et al.* (1986) and Milevsky (2006), recognize Mereu (1962) as a pioneer in providing means to compute annuity values under Gompertz-Makeham mortality law. However, Mereu himself also gave credit to McClintock (1874) who, few years after Makeham had proposed his mortality law (MAKEHAM, 1860, 1867), provided a formula to compute the expected present value of a continuous annuity in terms of the force of interest and Makeham's parameters. In turn, the method proposed by McClintock (1874) was already an attempt to improve the method developed by Makeham (1873), using only tables of ordinary gamma-functions to compute the actuarial value of a continuous annuity under Makeham law. The work of Rietz (1926) is an excellent reference for those interested in delving deeper into historical discussions about the application of calculus in actuarial mathematics, particularly on the introduction of continuous functions in the theory of annuities, starting from the seminal work by Woolhouse (1869).

Two decades after the publication of Mereu's work, Chan (1982) provided a closed-form expression for annuities under Gompertz-Makeham law based on the incomplete gamma function. Let us give a closer look at Chan's results and compare them to those of Castellares *et al.* (2020) regarding the scenario of a homogeneous population.

Following Chan (1982), under the Gompertz-Makeham law, the force of mortality at age x, μ_x, is defined as $\mu_x = A + BC^x$, and the probability of a person aged x survives to at least age $x+t$, ${}_tp_x$, is defined as ${}_tp_x = \exp\left(\frac{BC^x}{\ln C}\right).\exp(-At).\exp\left(\frac{-BC^{x+t}}{\ln C}\right)$. If $A = 0$, then we are under the Gompertz law. Moreover, letting δ be the force of interest per year, then, the expected present value of a whole life continuous annuity, subscribed by a person aged x, that pays \$1 per year as long as this person lives, $\bar{a}_x$, is given by $\bar{a}_x = \int_0^\infty e^{-\delta t}.{}_tp_x\,dt$. Using the incomplete gamma function $\Gamma(y,\alpha) = \int_\alpha^\infty e^{-v}.v^{y-1}dv$, and making $\alpha = \frac{BC^x}{\ln C}$, $y = -\frac{(A+\delta)}{\ln C}$ and $v = \frac{BC^{x+t}}{\ln C}$ substitutions, Chan (1982) proved that:

$$\bar{a}_x = \frac{1}{\ln C}.\exp\left(\frac{BC^x}{\ln C}\right).\left(\frac{BC^x}{\ln C}\right)^{\left(\frac{A+\delta}{\ln C}\right)}.\Gamma\left(-\frac{(A+\delta)}{\ln C},\frac{BC^x}{\ln C}\right) \tag{1}$$

Additionally, defining e_x as the complete life expectancy of a person aged x, then, when δ=0, $\bar{a}_x = e_x$. This well-known fact in the actuarial science implies that closed-form expressions derived to compute the expected present value of a continuous whole life annuity can also be applied to compute life expectancy.

In Castellares *et al.* (2020), the force of mortality under the Gompertz-Makeham law was defined as $\mu_x = c + ae^{bx}$, and in their Proposition 2, the authors proved that:

$$e_x = \frac{1}{b}.\exp\left(\frac{ae^{bx}}{b}\right).\left(\frac{ae^{bx}}{b}\right)^{c/b}.\Gamma\left(-\frac{c}{b},\frac{ae^{bx}}{b}\right) \tag{2}$$

Comparing expressions (1) and (2), when δ=0, it is easy to see that they are equal. To that end, we can simply make $A = c$, $B = a$ and $C = e^b$. Chan also provided a closed-form expression to the particular case where $A = 0$ (the Gompertz law). Therefore, the results rediscovered by Castellares *et al.* (2020) in their Proposition 1 (under Gompertz law) and Proposition 2 had been known in actuarial science literature for almost four decades.

At the end of the paper (in a brief paragraph added in proof) Chan also recognized Jan M. Hoem's warning that the result on the annuity value under Gompertz-Makeham law had already appeared (except for some notational differences) in a Danish textbook on insurance mathematics written by J. F. Steffensen in 1934. One year after the publication of Chan's work, Hoem himself published a paper highlighting lesser-known discoveries in actuarial mathematics by three Danish actuaries (Oppermann, Thiele and Gram). In his paper, the author states that Jørgen Pederson Gram, in 1904, published an article written in Danish where he used the incomplete gamma function to compute the actuarial value of an annuity under Gompertz-Makeham law (HOEM, 1983). Thus, albeit based on a secondary source, it seems that the use of the incomplete gamma function to compute the actuarial value of an annuity under Gompertz-Makeham law can be traced back to at least the work of Gram in the beginning of the 20th century.

Hoem (1983) argues that some of the early discoveries in actuarial mathematics, such as Gram's, fell into oblivion, even for the actuarial community, due to several causes, especially the fact that some articles were published in uncommon languages with very restricted circulation. However, towards the 1980s, after the rediscovery of Chan (1982), the annuity value and the expectation of life under a Gompertz-Makeham law started to be well known and used by the actuarial community.

The force of mortality under Gompertz-Makeham law can also be defined in terms of the modal age at death, M, as $\mu_x = \mu + \frac{e^{(x-M)/\beta}}{\beta}$ (MISSOV *et al.*, 2015; MILEVSKY, 2006). In this context, Milevsky (2006) also provided closed-form expressions to $\overline{a}_x$ and e_x (as well as to insurance factors, A_x) under Gompertz and Gompertz-Makeham laws. Under the Gompertz-Makeham law, Milevsky (2006) proved that:

$$e_x = \frac{\beta.\Gamma(-\mu\beta, \beta(\mu_x - \mu))}{e^{(M-x)\mu + \beta(\mu - \mu_x)}} \tag{3}$$

Thus, by making $c = \mu$, $b = \frac{1}{\beta}$ and $a = \frac{e^{-M/\beta}}{\beta}$, we can observe that expression (3) is equal to expression (2), reinforcing that those results were already well known and used by the actuarial science community. Nowadays, the work of Bowie (2021), which derives an analytical expression for annuities based on Makeham-Beard mortality law (RICHARDS, 2012), is a good example of the perpetuation of this practice in the field of actuarial science.

The case of a gamma-heterogeneous population

According to Pitacco (2004), even though early life tables implicitly assume homogeneity in populations, the problem of heterogeneity has always been a concern for actuaries, due to the negative effects adverse selection could have on the insurance and pension industries, and because ignoring heterogeneity could lead to an underestimation of the longevity risk. Furthermore, according to the author, the actuarial studies by Perks (1932), and later, Beard (1959), are precursors of the heterogeneity models.

Observation of improvements in life expectancy and the availability of better datasets on mortality (especially for advanced ages) put the overall validity of the Gompertz law into question since mortality increases appear to slow down (or stagnate) in advanced ages, rather than being constant as in the Gompertz model (BARBI *et al.*, 2018; BÖHNSTEDT; GAMPE, 2019; SALINARI; DE SANTIS, 2020; LI *et al.*, 2021). Formally, this evidence of mortality deceleration can be addressed with a multiplicative frailty model (VAUPEL; MANTON; STALLARD, 1979).

To deal with the problem of unobserved heterogeneity, a positive random variable (called frailty), Z, is introduced to modulate individual hazard (VAUPEL; MANTON; STALLARD, 1979). Therefore, in a heterogeneous population, the force of mortality conditional to $Z = z$ at a given age x is defined as $\mu(x \mid Z = z) = z\mu_x$. Due to mathematical and empirical reasons, it is typically assumed that the standard force of mortality, μ_x, follows the Gompertz law or the Gompertz-Makeham law and that Z is gamma-distributed, i.e., $Z \sim \Gamma(k,\lambda)$, with shape parameter $k > 0$ and scale parameter $\lambda > 0$ (VAUPEL; MANTON; STALLARD, 1979; BUTT; HABERMAN, 2004; MISSOV; FINKELSTEIN, 2011). At this point, it is worth mentioning that the combination of a Gompertz-Makeham mortality law with the gamma distribution implies in a cohort force of mortality (μ_x multiplied by the expected value of Z) belonging to Perks's family of survival models (PITACCO, 2004).

The survival function of the gamma-Gompertz-Makeham model is given by (CASTELLARES *et al.*, 2020):

$$S(x) = e^{-cx}\left[1 + \frac{a}{b\lambda}(e^{bx} - 1)\right]^{-k}, \tag{4}$$

and, consequently, $e_x = \int_0^\infty {}_tp_x\,dt = \int_0^\infty \frac{S(x+t)}{S(x)}dt$. Moreover, when $c = 0$, we are at the gamma--Gompertz setup.

In 2010, in a conference paper, Missov (2010) provided a correct closed form expression to life expectancy at birth (i.e., $x = 0$) under the gamma-Gompertz model.

$$e_0 = \frac{1}{bk}\,{}_2F_1\left(k, 1, k+1; 1 - \frac{a}{b\lambda}\right), \tag{5}$$

where, ${}_2F_1(p, q, m; w)$ is the ordinary hypergeometric function.

However, further attempts by Missov and colleagues (MISSOV, 2013; MISSOV; LENART, 2013) to provide a closed form expression to the remaining life expectancy for a person aged x (in both gamma-Gompertz and gamma-Gompertz-Makeham cases) led to incorrect results, and stimulated Castellares and colleagues (CASTELLARES *et al.*, 2020; CASTELLARES; PATRÍCIO; LEMONTE, 2020; PATRÍCIO, 2020) to provide valid expressions to compute e_x. Missov (2021) also wrote a letter, once aware of the work of Castellares and colleagues, revising his results about the correct formulation of e_x.

Therefore, in their study, Castellares *et al.* (2020), correctly generalized the original result proposed by Missov, providing a closed form expression to e_x in the gamma-Gompertz-Makeham model:

$$e_x = \frac{1}{bk+c}\ {}_2F_1\left(k, 1, k+1+\frac{c}{b};\ \frac{\left(1-\frac{a}{b\lambda}\right)}{\left(1-\frac{a}{b\lambda}+\frac{a}{b\lambda}e^{bx}\right)}\right) \tag{6}$$

Almost at the same time as the propositions of Missov and colleagues and subsequent revisions of Castellares and colleagues in the demographic field (and in related areas), Bemmaor and Glady (2012) applied the gamma-Gompertz setup to model customer lifetime. In their classic study, the authors also provided a closed-form expression to compute the mean customer lifetime in the gamma-Gompertz model. It was assumed that the customer lifetime model follows the Gompertz law, with cumulative distribution function defined by $F(x|\eta) = 1 - \exp(-\eta(e^{\theta x} - 1))$, where η indicates the customers' propensity to search for alternatives (BEMMAOR; GLADY, 2012). Additionally, it was also assumed that the parameter η was distributed gamma with shape parameter s and scale parameter B. Thus, the survival function could be defined as (BANTAN *et al.*, 2021):

$$S(x) = \frac{B^s}{(B+e^{\theta x}-1)^s} \tag{7}$$

In their gamma-Gompertz customer lifetime model, Bemmaor and Glady (2012) indicated that the mean customer lifetime – which, in the context of our discussion, could be interpreted as e_0 – is defined by:

$$e_0 = \frac{1}{\theta}.\frac{1}{s}.\ {}_2F_1\left(s, 1, s+1; \left(\frac{B-1}{B}\right)\right) \tag{8}$$

Therefore, for life expectancy at birth, and making $s = k$, $\theta = b$ and $\mathrm{B} = \frac{b\lambda}{a}$, we can observe that expression (8) is equal to expression (5), which shows that different academic fields were working on the same problem in a parallel manner and independently providing their contributions. However, for the case in which the conditional mean lifetime given customer i is alive at age x, Bemmaor and Glady also provided an incorrect expression to e_x.

More recently Adler (2022) revised some results in Bemmaor and Glady (2012), in particular, providing (independently) a correct expression to e_x under the gamma-Gompertz model:

$$e_x = \frac{1}{\theta}.\frac{1}{s}.\ {}_2F_1\left(s, 1, s+1; \left(\frac{B-1}{B+e^{\theta x}-1}\right)\right) \tag{9}$$

Using the same reparametrization presented, it can be observed that (for $c = 0$) equations (6) and (9) are equal, reinforcing the result previously provided by Missov (2010) and later extended/revised by Castellares *et al.* (2020) were also independently rediscovered in the management field by Bemmaor and Glady (2012) and Adler (2022). Therefore, this comparison highlights that discussions that started in the actuarial field are still drawing the attention of researchers in multiple areas.

Final remarks

To conclude, it is worth noting that this note is not intended as an authoritative discussion of who invented what, especially because the development of closed form expressions to value annuities (and life expectancy) under Gompertz-Makeham law has received (and is still receiving) contributions from many eminent researchers, and there may be other unknown results and authors patiently waiting for their deserved recognition.

Moreover, as already highlighted by Pitacco (2004), throughout history, several developments in actuarial science were ignored (and independently rediscovered) by demographers and researchers from other fields, and vice versa. In this regard, Olshansky (2010) warns researchers about the importance of avoiding a myopic discipline-view of the issues faced and emphasizes the benefits brought about by interdisciplinarity in science.

Therefore, with these ideas in mind, this note humbly intended to acknowledge previous results established in other fields, particularly in actuarial science, in the hope that it could also stimulate interdisciplinarity and provide the *background* for further developments, especially since the derivation of closed-form expressions for life expectancy (and annuities) based on particular mortality laws is an interest common to multiple fields such as Actuarial Science, Biology, Demography, Statistics, among others.

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Resumo

Expressões de forma fechada para as expectativas de vida de Gompertz-Makeham: uma nota histórica

Resultados bem conhecidos pela comunidade atuarial sobre expressões de forma fechada para esperança de vida de Gompertz e Gompertz-Makeham para uma pessoa de idade x ainda estão sendo redescobertos de forma independente nos dias atuais. Esta nota visa fornecer algum reconhecimento aos resultados anteriores sobre expressões de forma fechada para expectativa de vida de Gompertz e Gompertz-Makeham, especialmente no campo das ciências atuariais, na esperança de estimular a interdisciplinaridade e fornecer o pano de fundo para novos desenvolvimentos, em especial porque a derivação de expressões de forma fechadas para expectativa de vida (e anuidades) com base em leis de mortalidade despertam o interesse de várias áreas, como ciências atuariais, biologia, demografia, estatística, entre outras.

Palavras-chave: Lei de mortalidade de Gompertz-Makeham. Expectativa de vida. Ciências atuariais. Anuidades. Fragilidade. Modelo gama-Gompertz.

Resumen

Expresiones de forma cerrada a las esperanzas de vida de Gompertz-Makeham: una nota histórica

Los resultados bien conocidos por la comunidad actuarial sobre las expresiones de forma cerrada de las esperanzas de vida de Gompertz y Gompertz-Makeham para una persona de edad x todavía se están redescubriendo de forma independiente en la actualidad. Esta nota pretende reconocer algunos resultados anteriores sobre expresiones cerradas para la esperanza de vida de Gompertz y Gompertz-Makeham, en especial en el campo de las ciencias actuariales, con la esperanza de fomentar la interdisciplinariedad y proporcionar el telón de fondo para futuros desarrollos, sobre todo desde que la derivación de expresiones cerradas para la esperanza de vida (y anualidades) basado en leyes de mortalidad despertó el interés de varias áreas, como las ciencias actuariales, biología, demografía, estadística, entre otras.

Palabras clave: Ley de mortalidad de Gompertz-Makeham. Expectativa de vida. Ciencias actuariales. Anualidades. Fragilidad. Modelo gamma-Gompertz.

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